RING MAPS AND THE BOUSFIELD LATTICE

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Abstract. Given a map of commutative rings $f : R \to S$, extension by scalars and the forgetful functor give an induced adjoint pair of functors between the derived categories $D(R)$ and $D(S)$. We use these functors to relate the Bousfield lattices, and thick and localizing subcategories, of $D(R)$ and $D(S)$. One consequence is new information about the structure of derived categories of non-Noetherian rings.

1. Introduction

A commutative ring map $f : R \to S$ induces a functor on module categories $f_* : \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}S$, where $f_*(M) = M \otimes_R S$. This induces a functor $Ch(R) \to Ch(S)$ on complexes. Let $f_\bullet : D(R) \to D(S)$ be the derived functor $f_\bullet = Lf_*$, and let $i_\bullet = Li : D(S) \to D(R)$ be its adjoint, induced by the forgetful functor $i : \mathsf{Mod}\text{-}S \to \mathsf{Mod}\text{-}R$.

We approach these classical algebraic objects from the perspective of algebraic topology and tensor-triangulated category theory. So we think of the unbounded derived category $D(T)$ of a commutative ring T as a monogenic stable homotopy category [HPS97]. It has a symmetric monoidal tensor product, $- \otimes^L_T -$, which we will denote as the smash product $- \wedge -$. The unit of this smash product, the module T concentrated in degree zero, is the sphere object - it is a small, weak generator. We have arbitrary coproducts, given by degree-wise direct sums, and Brown representability holds.

A triangulated subcategory of $D(T)$ is called *thick* if it is closed under retracts. An object is called *finite* if it is in the thick subcategory generated by the sphere object. A triangulated subcategory is called *localizing* if it is closed under arbitrary coproducts. We are interested in characterizing the localizing subcategories, and the thick subcategories of finite objects, in $D(T)$. When the ring T is Noetherian, this has been done [Nee92]: these are classified by subsets, and specialization-closed subsets, respectively, of the prime spectrum $\mathsf{Spec}\, T$. Furthermore, Thomason [Tho97] showed the thick subcategories of finite objects of an arbitrary commutative ring can be classified by certain subsets of $\mathsf{Spec}\, T$. More broadly, much work has been done towards understanding thick subcategories of finite objects in general triangulated and tensor-triangulated categories. See, e.g. [B05, BIK11].

However, localizing subcategories seem harder to pin down. One approach is to study Bousfield classes. The *Bousfield class* of an object X is $\{W \mid W \wedge X = 0\}$, and

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every Bousfield class is a localizing subcategory. When there is a set of Bousfield classes, these form a complete lattice called the *Bousfield lattice*. We have some understanding of the Bousfield lattice of the stable homotopy category of spectra [HP99], of the derived category of a particular non-Noetherian ring Λ [DP08], and of a general tensor-triangulated category [IK11]. The Bousfield lattice of the derived category $D(T)$ of a Noetherian ring T is more or less completely understood - every localizing subcategory is a Bousfield class, and the Bousfield lattice is in bijection with subsets of $\mathsf{Spec}\, T$.

In this paper, in order to better understand the Bousfield lattice and localizing subcategories in the derived category of an arbitrary commutative ring, we try to relate the derived categories of different rings. So we start with a ring map $f : R \to S$, and work with $f_\bullet : D(R) \to D(S)$ and $i_\bullet : D(S) \to D(R)$ as defined above. Placing various hypotheses on the map f gives a range of results.

For example, there are two interesting well-known sublattices DL and BA within the Bousfield lattice BL, and we have (Props. 2.6 and 4.6, Lemma 4.5)

Proposition 1.1. *$f_\bullet$ and $i_\bullet$ define order-preserving operations between the Bousfield lattices of $D(R)$ and $D(S)$. The map $f_\bullet$ sends $\mathsf{DL}_{D(R)}$ to $\mathsf{DL}_{D(S)}$ and $\mathsf{BA}_{D(R)}$ to $\mathsf{BA}_{D(S)}$.*

Proposition 1.2. *Assume $f_\bullet i_\bullet \langle X\rangle = \langle X\rangle$ for all X. Then $i_\bullet$ injects $\mathsf{DL}_{D(S)}$ into $\mathsf{DL}_{D(R)}$ and $\mathsf{BA}_{D(S)}$ into $\mathsf{BA}_{D(R)}$.*

We also define a quotient lattice $\mathsf{BL}_{D(R)}/J$ of the Bousfield lattice of $D(R)$, and show (Prop. 4.2) that when $f_\bullet i_\bullet \langle X\rangle = \langle X\rangle$ for all X, $f_\bullet$ induces an isomorphism

$$\overline{f_\bullet} : \mathsf{BL}_{D(R)}/J \xrightarrow{\cong} \mathsf{BL}_{D(S)},$$

with inverse $\underline{i_\bullet}$. Section 6.3 shows that when R and S are Noetherian and f is surjective, $\overline{f_\bullet}$ induces a complete splitting $\mathsf{BL}_{D(R)} \cong \mathsf{BL}_{D(S)} \times J$.

In Sections 5 and 6, we show that $f_\bullet$ and $i_\bullet$ give well-defined operations on thick and localizing subcategories, and, using the notion of support, connect with the classification theorems mentioned above.

For example, when $f : R \to S$ is surjective, and S is Noetherian, let $T = f^{-1}(\mathsf{Spec}\, S) \subseteq \mathsf{Spec}\, R$. We have the following (Props. 5.12 and 5.21).

Proposition 1.3. *Let X be an element of $D(S)$, or a thick subcategory of $D(S)$. Let Y be an element of $D(R)$, or a thick subcategory of $D(R)$. Then*

$$supp(i_\bullet X) = f^{-1}(supp(X)), \text{ and } f^{-1}(supp(f_\bullet Y)) \subseteq supp(Y) \cap T.$$

When R is also Noetherian, the latter inclusion is an equality, and the two statements hold with X or Y localizing subcategories.

Also, when $f : R \to S$ is surjective, and S is Noetherian, we define finite objects $R/\tilde{\mathfrak{p}}$ in $D(R)$ that mimic the Koszul objects $S/\mathfrak{p}$ in $D(S)$. These allow us to explicitly pull back structure from $D(S)$ to $D(R)$.

The final section gives an application of these results to a map $f : \Gamma \to \Lambda$ between two non-Noetherian rings, where we are also able to demonstrate a complete splitting of Bousfield lattices.

2. General $f : R \to S$

In this section, let $f : R \to S$ be any ring homomorphism, and $f_* : \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}S$ as above. Let $f_\bullet$ be the left derived functor $f_\bullet = Lf_* = L(- \otimes_R S) : D(R) \to D(S)$. Then [HPS97, 9.3.1] shows that $f_\bullet$ is a stable morphism - it is exact, has $f_\bullet(R) = S$, and $f_\bullet(X \wedge Y) = f_\bullet X \wedge f_\bullet Y$. As a left adjoint, it commutes with coproducts and sends projectives to projectives.

The right adjoint of $f_\bullet$ is $i_\bullet = Li : D(S) \to D(R)$, induced by the forgetful functor $i : \mathsf{Mod}\text{-}S \to \mathsf{Mod}\text{-}R$. It is exact, commutes with coproducts and products, and sends injectives to injectives. It is injective in the sense that $i_\bullet(X) = 0$ implies $X = 0$, simply because an acyclic complex of S-modules is acyclic whether we think of it as a complex of R-modules or S-modules. The adjointness means

$$\mathrm{Hom}^*_{D(S)}(f_\bullet X, Y) \cong \mathrm{Hom}^*_{D(R)}(X, i_\bullet Y).$$

The following lemma will be used frequently.

Lemma 2.1. *For all objects A in $D(R)$ and B in $D(S)$, we have*

$$i_\bullet(f_\bullet A \wedge B) = A \wedge i_\bullet B.$$

Proof. This is the projection formula in [Wei94, 10.8.5]. □

Remark 2.2. Take $z \in R = [R, R]_*$, and consider the morphism $R \xrightarrow{z} R$ in $D(R)$. Applying $f_\bullet$ to this, we get

$$\left(f_\bullet(R) \xrightarrow{f_\bullet(z)} f_\bullet(R)\right) = \left(R \otimes_R S \xrightarrow{z\otimes 1} R \otimes_R S\right)$$
$$= \left(R \otimes_R S \xrightarrow{1\otimes f(z)} R \otimes_R S\right) = \left(S \xrightarrow{f(z)} S\right).$$

From this we conclude the following.

Lemma 2.3. *The functor $f_\bullet$ takes finite objects to finite objects.*

2.1. Ideals and prime ideals.

Here we introduce two important classes of objects. For any finitely generated ideal $\mathfrak{r} = (z_1, ..., z_n)$ in a ring T, let $T/\mathfrak{r}$ denote the wedge $T/z_1 \wedge T/z_2 \wedge \cdots \wedge T/z_n$, where T/z_i is the cofiber of the map $T \xrightarrow{z_i} T$. These are called *Koszul objects* in [BIK11].

Lemma 2.4. *Given two finitely generated ideals $\mathfrak{r}$, $\mathfrak{t}$ in a ring T, if $\mathfrak{r} \subseteq \mathfrak{t}$ then $T/\mathfrak{t} \in \mathsf{th}(T/\mathfrak{r})$. Therefore different choices of generators of an ideal $\mathfrak{r}$ will generate the same thick subcategory $\mathsf{th}(T/\mathfrak{r})$.*

Proof. This is basically Lemma 6.0.9 in [HPS97]. The proof there requires the ideals be finitely generated, but not prime. □

For a prime ideal $\mathfrak{p}$ in a ring T, let $T_\mathfrak{p}$ be the localization at $\mathfrak{p}$. Then let $k_\mathfrak{p} = T_\mathfrak{p}/\mathfrak{p}T_\mathfrak{p}$ be the residue field of $\mathfrak{p}$; let $\overline{k_\mathfrak{p}}$ be this field thought of in $D(T)$. Then [HPS97, 3.7.2] shows that $\overline{k_\mathfrak{p}}$ is a skew field object in $D(T)$.

For every prime ideal $\mathfrak{p}$, the object $\overline{k_{\mathfrak{p}}} \wedge \overline{k_{\mathfrak{p}}}$ is nonzero, because it has homology $\mathrm{Ext}(k_{\mathfrak{p}}, k_{\mathfrak{p}}) \neq 0$.

Lemma 2.5. *Let $\mathfrak{q}$, $\mathfrak{p}$ be prime ideals of a ring T. If $\mathfrak{p} \neq \mathfrak{q}$, then $\overline{k_{\mathfrak{p}}} \wedge \overline{k_{\mathfrak{q}}} = 0$.*

Proof. Without loss of generality, take $r \in \mathfrak{p} \backslash \mathfrak{q}$. Then since $k_{\mathfrak{q}}$ is $\mathfrak{q}$-local, and $r \notin \mathfrak{q}$, the map $k_{\mathfrak{q}} \xrightarrow{r} k_{\mathfrak{q}}$ is an isomorphism, and induces an equivalence in $D(T)$. On the other hand, $k_{\mathfrak{p}}$ is $\mathfrak{p}$-torsion, so $k_{\mathfrak{p}} \xrightarrow{r} k_{\mathfrak{p}}$ is nilpotent (some power of it is zero). Since

$$\overline{k_{\mathfrak{p}}} \wedge \overline{k_{\mathfrak{q}}} \xrightarrow{1 \wedge r} \overline{k_{\mathfrak{p}}} \wedge \overline{k_{\mathfrak{q}}} \quad \text{is an equivalence, and}$$

$$\overline{k_{\mathfrak{p}}} \wedge \overline{k_{\mathfrak{q}}} \xrightarrow{r \wedge 1} \overline{k_{\mathfrak{p}}} \wedge \overline{k_{\mathfrak{q}}} \quad \text{is nilpotent,}$$

we must have $\overline{k_{\mathfrak{p}}} \wedge \overline{k_{\mathfrak{q}}} = 0$. □

2.2. **Bousfield lattice.** Recall that, given an object X, the *Bousfield class* of X is defined to be

$$\langle X \rangle := \{W \mid X \wedge W = 0\}.$$

Each Bousfield class is a localizing subcategory. The collection of Bousfield classes has a partial ordering, based on reverse inclusion, so

$$\langle X \rangle \leq \langle Y \rangle \text{ if and only if } W \wedge Y = 0 \text{ implies } W \wedge X = 0.$$

Then $\langle 0 \rangle$ is a minimum, and the Bousfield class of the sphere object is a maximum. The coproduct gives a join operation. For more on Bousfield classes, see [Bou79a] or [HP99].

Iyengar and Krause showed recently that the collection of homological Bousfield classes in the derived category of *any* ring R is a set [IK11, Thm. 3.1]. Thus for every commutative ring T, the collection of Bousfield classes forms a complete *Bousfield lattice*, denoted $\mathsf{BL}_{D(T)}$. The functors $f_\bullet$ and $i_\bullet$ induce maps between the Bousfield lattices of $D(R)$ and $D(S)$. If we consider a Bousfield class $\langle X \rangle$ as a localizing subcategory, then we can map this to $f_\bullet(\langle X \rangle)$ as a subcollection in $D(S)$. However, in general $f_\bullet(\langle X \rangle)$ will not be triangulated, because $f_\bullet \circ i_\bullet \neq 1_{D(S)}$.

So instead we define an operation $f_\bullet : \mathsf{BL}_{D(R)} \to \mathsf{BL}_{D(S)}$ by $\langle X \rangle \mapsto \langle f_\bullet X \rangle$. We also define an operation $i_\bullet : \mathsf{BL}_{D(S)} \to \mathsf{BL}_{D(R)}$ by $\langle X \rangle \mapsto \langle i_\bullet X \rangle$. For the rest of this paper, $f_\bullet \langle X \rangle$ and $i_\bullet \langle X \rangle$ will mean $\langle f_\bullet X \rangle$ and $\langle i_\bullet X \rangle$.

Proposition 2.6. *Both $f_\bullet$ and $i_\bullet$, as defined above, are well-defined, order-preserving operations on Bousfield lattices, and both preserve arbitrary joins.*

Proof. First we show that $\langle Y \rangle \leq \langle X \rangle$ implies $\langle i_\bullet Y \rangle \leq \langle i_\bullet X \rangle$. Suppose $\langle Y \rangle \leq \langle X \rangle$ and $W \wedge i_\bullet X = 0$. Then Lemma 2.1 implies $i_\bullet(f_\bullet W \wedge X) = 0$, so $f_\bullet W \wedge X = 0$. Thus $f_\bullet W \wedge Y = 0$, and $W \wedge i_\bullet Y = 0$.

This implies that if $\langle Y \rangle = \langle X \rangle$, then $\langle i_\bullet Y \rangle = \langle i_\bullet X \rangle$, so $i_\bullet(-)$ is well-defined and order-preserving.

Now suppose $\langle Y \rangle \leq \langle X \rangle$ and $W \wedge f_\bullet X = 0$. Then from the lemma $X \wedge i_\bullet W = 0$, so $Y \wedge i_\bullet W = 0$, which implies $f_\bullet Y \wedge W = 0$. Therefore $f_\bullet(-)$ is order-preserving and well-defined.

It's clear that $f_\bullet$ and $i_\bullet$ preserve arbitrary joins. □

As an aside, note that this implies $\langle i_\bullet X \rangle \leq \langle i_\bullet S \rangle$ for all X in $D(S)$.

Let J be the image of $Kerf_\bullet$ in $\mathsf{BL}_{D(R)}$, in other words

$$J := \{\langle X\rangle \mid f_\bullet\langle X\rangle = \langle 0\rangle\}.$$

Proposition 2.7. *The subposet J is a complete principal ideal in $\mathsf{BL}_{D(R)}$, and $f_\bullet$ induces a poset map*

$$\overline{f_\bullet} : \mathsf{BL}_{D(R)}/J \to \mathsf{BL}_{D(S)}.$$

Proof. Suppose $\langle Y\rangle \leq \langle X\rangle$ and $\langle f_\bullet X\rangle = \langle 0\rangle$. Then $\langle f_\bullet Y\rangle \leq \langle f_\bullet X\rangle$, so $\langle f_\bullet Y\rangle = \langle 0\rangle$ and J is a lattice ideal. It is complete because it is closed under arbitrary joins. If we define

$$\langle M\rangle := \bigvee_{\langle Y\rangle \in J} \langle Y\rangle,$$

then $J = \{\langle X\rangle \mid \langle X\rangle \leq \langle M\rangle\}$ is principal.

This is not enough to guarantee an induced map on the quotient lattice (see [HP99, 3.11]). For this, we also need to know that if $\langle X\rangle \equiv \langle Y\rangle (\mathrm{mod}\ J)$, then $f_\bullet\langle X\rangle = f_\bullet\langle Y\rangle$. As in [HP99, Sect. 3], $\langle X\rangle$ and $\langle Y\rangle$ are equivalent if and only if $\langle X\rangle \vee \langle M\rangle = \langle Y\rangle \vee \langle M\rangle$. But then since $\langle f_\bullet M\rangle = \langle 0\rangle$,

$$\langle f_\bullet X\rangle = \langle f_\bullet X\rangle \vee \langle f_\bullet M\rangle = f_\bullet(\langle X\rangle \vee \langle M\rangle) = \langle f_\bullet Y\rangle \vee \langle f_\bullet M\rangle = \langle f_\bullet Y\rangle.$$

□

For brevity, we will denote $\mathsf{BL}_{D(R)}/J$ by BL/J. Note that in BL/J we have $\overline{f_\bullet}\langle X\rangle = \langle 0\rangle$ if and only if $\langle X\rangle = \langle 0\rangle$.

Lemma 2.8. *In $\mathsf{BL}_{D(R)}$ we have $i_\bullet \circ f_\bullet\langle X\rangle \leq \langle X\rangle$, and in BL/J we have $i_\bullet \circ \overline{f_\bullet}\langle X\rangle = \langle X\rangle$, for all $\langle X\rangle$. Thus in BL/J, if $\overline{f_\bullet}\langle X\rangle = \overline{f_\bullet}\langle Y\rangle$, then $\langle X\rangle = \langle Y\rangle$.*

Proof. An object W has $W \wedge i_\bullet f_\bullet Y = 0$ iff $f_\bullet W \wedge f_\bullet Y = 0$ iff $f_\bullet(W \wedge Y) = 0$.

Therefore $W \wedge Y = 0$ implies $W \wedge (i_\bullet f_\bullet Y) = 0$. And in BL/J the converse holds as well.

If $\langle \overline{f_\bullet} X\rangle = \langle \overline{f_\bullet} Y\rangle$, then $\langle X\rangle = \langle i_\bullet \overline{f_\bullet} X\rangle = \langle i_\bullet \overline{f_\bullet} Y\rangle = \langle Y\rangle$. □

2.3. BA **and** DL**.** Here we begin an investigation of the effects of $f_\bullet$ and $i_\bullet$ on the sublattices BA and DL of BL. See also Section 4.1, and [Bou79a, HP99, DP08, IK11] for more results on BA and DL.

The distributive lattice DL is defined to be the set of Bousfield classes $\langle X\rangle$ such that $\langle X\rangle = \langle X \wedge X\rangle$. In DL, the meet of $\langle X\rangle$ and $\langle Y\rangle$ is given by $\langle X \wedge Y\rangle$, so the meet distributes over arbitrary joins, hence the name.

Because we know there is a set of Bousfield classes, we can define a complementation operator $a(-)$ on Bousfield classes, by

$$a\langle X\rangle = \bigvee_{\langle X\rangle \wedge \langle Y\rangle = \langle 0\rangle} \langle Y\rangle.$$

The Boolean algebra BA is defined to be the set of complemented Bousfield classes; i.e. all $\langle X\rangle$ such that there exists a class $\langle X^c\rangle$ with $\langle X\rangle \wedge \langle X^c\rangle = \langle 0\rangle$ and $\langle X\rangle \vee \langle X^c\rangle$ the maximum Bousfield class. It is not hard to show that if $\langle X\rangle$ is complemented, then $\langle X^c\rangle = a\langle X\rangle$.

In general, $\mathsf{BA} \subseteq \mathsf{DL}$. For every finite W, $\langle W \rangle$ is in BA. For every smashing localization functor $L : D(T) \to D(T)$, the acyclics $\langle LT \rangle$ are in BA.

Proposition 2.9. *$f_\bullet$ maps $\mathsf{DL}_{D(R)}$ into $\mathsf{DL}_{D(S)}$, and $\mathsf{BA}_{D(R)}$ into $\mathsf{BA}_{D(S)}$. If $\langle X \rangle$ in $\mathsf{BA}_{D(R)}$ has complement $\langle X^c \rangle$, then $\langle f_\bullet X \rangle$ has complement $\langle f_\bullet(X^c) \rangle$.*

Proof. If $\langle Y \rangle = \langle Y \wedge Y \rangle$, then $\langle f_\bullet Y \rangle = \langle f_\bullet Y \wedge f_\bullet Y \rangle$.

If $\langle X \rangle$ has $\langle X \rangle \vee \langle X^c \rangle = \langle R \rangle$ and $\langle X \rangle \wedge \langle X^c \rangle = \langle 0 \rangle$, then $\langle f_\bullet X \rangle \vee \langle f_\bullet X^c \rangle = \langle f_\bullet R \rangle = \langle S \rangle$ and $\langle f_\bullet X \rangle \wedge \langle f_\bullet X^c \rangle = \langle 0 \rangle$, so $\langle (f_\bullet X)^c \rangle = \langle f_\bullet X^c \rangle$. □

3. Surjective $f : R \to S$

In this section, let $f : R \to S$ be a surjective ring map.

Lemma 3.1. *On the module level, we have $f_* \circ i = Id_{\underline{\mathsf{Mod}-S}}$.*

Proof. The S-module map $\phi : N \to i(N) \otimes_R S$ defined by $\phi(n) = n \otimes 1$ has inverse $\psi(n \otimes s) = (n.s)$. □

However, as noted above, on derived categories we don't necessarily have $f_\bullet \circ i_\bullet = 1_{D(S)}$. This is because i doesn't send projectives to projectives, so $L(f_* \circ i) \neq L(f_*) \circ Li$ (see [Wei94, 10.8.2]).

Lemma 3.2. *The map $f^{-1} : \mathsf{Spec}\, S \to \mathsf{Spec}\, R$ is injective.*

Proof. $\mathfrak{p} = f(f^{-1}(\mathfrak{p})) = f(f^{-1}(\mathfrak{q})) = \mathfrak{q}$. □

In fact, as a map of topological spaces, f^{-1} is a homeomorphism onto its image [H77, ex.2.18].

Lemma 3.3. *Let $\mathfrak{p} \subseteq S$ be a prime ideal. Then $f_\bullet(R_{f^{-1}\mathfrak{p}}) = S_\mathfrak{p}$ as objects in $D(S)$.*

Proof. First note that $R_{f^{-1}\mathfrak{p}}$ is flat over R, so

$$f_\bullet(R_{f^{-1}\mathfrak{p}}) = L(- \otimes_R S)(R_{f^{-1}\mathfrak{p}}) = R_{f^{-1}\mathfrak{p}} \otimes_R S.$$

We have an S-module map $\phi : R_{f^{-1}\mathfrak{p}} \otimes_R S \longrightarrow S_\mathfrak{p}$ given by

$$\phi\left(\frac{a}{b} \otimes c\right) = \frac{f(a)c}{f(b)}.$$

If $\frac{f(a)c}{f(b)} = 0$ then $f(a)c = 0$, so $\frac{a}{b} \otimes c = \frac{1}{b} \otimes f(a)c = 0$.

Given $\frac{y}{z} \in S_\mathfrak{p}$, with $z \in S \backslash \mathfrak{p}$, since f is surjective we have $b \in R \backslash f^{-1}\mathfrak{p}$ such that $f(b) = z$. Then

$$\phi\left(\frac{1}{b} \otimes y\right) = \frac{y}{f(b)} = \frac{y}{z}.$$

□

The next proposition plays an important role in later computations.

Proposition 3.4. *Let $\mathfrak{p} \subseteq S$ be a prime ideal. Then $f_\bullet(\overline{k_{f^{-1}\mathfrak{p}}}) = \overline{k_\mathfrak{p}}$.*

Proof. Define $\mathfrak{q} = f^{-1}\mathfrak{p}$, and choose generators y_i so $(y_1, y_2, ...) = \mathfrak{q}$. Define $z_i = f(y_i) \in \mathfrak{p}$ and note that $(z_1, z_2, ...) = \mathfrak{p}$.

The R-module $k_\mathfrak{q} = R_\mathfrak{q}/\mathfrak{q}R_\mathfrak{q}$ is $\mathfrak{q}$-local, and an $R_\mathfrak{q}$-module. In $D(R_\mathfrak{q})$, we have the object $\overline{k_\mathfrak{q}}$ concentrated in degree zero. In local rings, projective modules are free, and we have a projective resolution of $k_\mathfrak{q}$ in $\mathsf{Mod}\text{-}R_\mathfrak{q}$:

$$\cdots \longrightarrow \bigoplus R_\mathfrak{q} \longrightarrow \bigoplus R_\mathfrak{q} \xrightarrow{\oplus y_i} R_\mathfrak{q} \longrightarrow 0,$$

where the direct sums are infinite if $(y_1, y_2, ...) = \mathfrak{q}$ is not finitely generated. This is a representation of $\overline{k_\mathfrak{q}}$ in $D(R_\mathfrak{q})$.

Now $R_\mathfrak{q}$ is a flat R-module, so this resolution is also a flat resolution of $k_\mathfrak{q}$ in $\mathsf{Mod}\text{-}R$, so represents $\overline{k_\mathfrak{q}}$ in $D(R)$. To compute $f_\bullet(\overline{k_\mathfrak{q}})$, we just apply $- \otimes_R S$ to this resolution. In Lemma 3.3 we saw that as S-modules, $R_\mathfrak{q} \otimes_R S = S_\mathfrak{p}$, and $y_i \otimes 1 = f(y_i) = z_i$. So we get that $f_\bullet(\overline{k_\mathfrak{q}})$ is represented by

$$\cdots \longrightarrow \bigoplus S_\mathfrak{p} \longrightarrow \bigoplus S_\mathfrak{p} \xrightarrow{\oplus z_i} S_\mathfrak{p} \longrightarrow 0.$$

This shows that $f_\bullet(\overline{k_\mathfrak{q}})$ has nonzero homology only in degree zero, and its zeroth homology is $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} = k_\mathfrak{p}$.

The object $\overline{k_\mathfrak{p}}$ in $D(S)$ can also be represented as a bounded-below chain complex of projectives, with nonzero homology only in degree zero. Since $\overline{k_\mathfrak{p}}$ and $f_\bullet(\overline{k_\mathfrak{q}})$ are both equivalent to bounded-below complexes of projectives, with the same zeroth homology and zero homology elsewhere, there is a weak equivalence between them. $\square$

4. A MAP $f : R \to S$ SUCH THAT $f_\bullet i_\bullet\langle X\rangle = \langle X\rangle$ FOR ALL X

This happens, for example, when f is surjective and S is Noetherian (see Section 5). It also happens with the specific non-Noetherian projection $f : \Gamma \to \Lambda$ that we examine in Section 7.

Note that $f_\bullet i_\bullet\langle X\rangle = \langle X\rangle$ means $i_\bullet(-)$ is injective in the sense that $i_\bullet\langle X\rangle = i_\bullet\langle Y\rangle$ implies $\langle X\rangle = \langle Y\rangle$.

Lemma 4.1. *The following are equivalent:*

1. $f_\bullet i_\bullet\langle X\rangle = \langle X\rangle$ *for all* $\langle X\rangle$
2. $i_\bullet W \wedge i_\bullet X = 0$ *if and only if* $i_\bullet(W \wedge X) = 0$
3. $i_\bullet\langle Y \wedge X\rangle = \langle i_\bullet Y\rangle \wedge \langle i_\bullet X\rangle$.

Proof. For (1) $\Leftrightarrow$ (2), note that $W \wedge f_\bullet i_\bullet X = 0$ iff $i_\bullet W \wedge i_\bullet X = 0$, and $W \wedge X = 0$ iff $i_\bullet(W \wedge X) = 0$.

For (1) $\Leftrightarrow$ (3), note that $W \wedge i_\bullet(Y \wedge X) = 0$ iff $f_\bullet W \wedge (Y \wedge X) = 0$ iff $(f_\bullet W \wedge Y) \wedge X = 0$, and $W \wedge i_\bullet X \wedge i_\bullet Y = 0$ iff $(f_\bullet W \wedge f_\bullet i_\bullet X) \wedge Y = 0$ iff $(f_\bullet W \wedge Y) \wedge (f_\bullet i_\bullet X) = 0$. $\square$

Proposition 4.2. *If $f_\bullet i_\bullet\langle X\rangle = \langle X\rangle$ for all $\langle X\rangle$, then $f_\bullet : \mathsf{BL}_{D(R)} \to \mathsf{BL}_{D(S)}$ is onto, and we have a bijection of posets*

$$\overline{f_\bullet} : \mathsf{BL}_{D(R)}/J \xrightarrow{\cong} \mathsf{BL}_{D(S)},$$

where $i_\bullet(-)$ is the inverse.

Proof. It's clear that $f_\bullet$ and $\overline{f_\bullet}$ are onto. In BL/J we have $\overline{f_\bullet}\langle X\rangle = \langle 0\rangle$ if and only if $\langle X\rangle = \langle 0\rangle$. $\square$

4.1. **Poset adjoints.** As a poset map, because $i_\bullet(-)$ preserves joins on $\mathsf{BL}_{D(S)}$, it has a poset map right adjoint $r : \mathsf{BL}_{D(R)} \to \mathsf{BL}_{D(S)}$, see [HP99, 3.5]. We know

$$r\langle Y\rangle = \bigvee \{\langle X\rangle \mid i_\bullet\langle X\rangle \leq \langle Y\rangle\}, \text{ and}$$

$$i_\bullet\langle X\rangle \leq \langle Y\rangle \text{ if and only if } \langle X\rangle \leq r\langle Y\rangle.$$

Proposition 4.3. *If $f_\bullet i_\bullet\langle X\rangle = \langle X\rangle$ for all $\langle X\rangle$, then $f_\bullet\langle X\rangle = r\langle X\rangle$ for all $\langle X\rangle$, so*

$$\langle i_\bullet X\rangle \leq \langle Y\rangle \text{ if and only if } \langle X\rangle \leq \langle f_\bullet Y\rangle.$$

Proof. Lemma 2.8 implies that $f_\bullet\langle X\rangle \leq r\langle X\rangle$ for all $\langle X\rangle$. For the other direction, it suffices to show that if $\langle i_\bullet X\rangle \leq \langle Y\rangle$, then $\langle X\rangle \leq \langle f_\bullet Y\rangle$.

If $\langle i_\bullet X\rangle \leq \langle Y\rangle$ and $W \wedge f_\bullet Y = 0$, then Lemma 2.1 implies $i_\bullet W \wedge Y = 0$, so $i_\bullet W \wedge i_\bullet X = 0$. It follows from Lemma 4.1 that $i_\bullet(W \wedge X) = 0$, so $W \wedge X = 0$. $\square$

The BL operation $f_\bullet$ also preserves arbitrary joins, so has a poset map right adjoint. On the object level, we know that $f_\bullet$ is left adjoint to $i_\bullet$, and so it is natural to ask if $i_\bullet$ is the poset adjoint of $f_\bullet$.

Proposition 4.4. *Assume $f_\bullet i_\bullet\langle X\rangle = \langle X\rangle$ for all X. Then on the level of Bousfield classes, we have*

$$\langle f_\bullet X\rangle \leq \langle Y\rangle \Leftarrow \langle X\rangle \leq \langle i_\bullet Y\rangle,$$

but the forward direction need not hold. In the quotient $\overline{f_\bullet} : \mathsf{BL}/J \to \mathsf{BL}$ we do indeed have

$$\overline{f_\bullet}\langle X\rangle \leq \langle Y\rangle \Leftrightarrow \langle X\rangle \leq i_\bullet\langle Y\rangle,$$

so $\overline{f_\bullet}$ and $i_\bullet(-)$ are poset adjoints.

Proof. First suppose $\langle X\rangle \leq \langle i_\bullet Y\rangle$ and $W \wedge Y = 0$. Then $i_\bullet(W \wedge Y) = 0$, which using the above lemma means $i_\bullet W \wedge i_\bullet Y = 0$, so $i_\bullet W \wedge X = 0$, and $W \wedge f_\bullet X = 0$.

On the other hand, suppose $\langle f_\bullet X\rangle \leq \langle Y\rangle$ and $W \wedge i_\bullet Y = 0$. Then $f_\bullet W \wedge Y = 0$, $f_\bullet W \wedge f_\bullet X = 0$, and $f_\bullet(W \wedge X) = 0$. At the BL level, this does not necessarily mean $W \wedge X = 0$. In the quotient, however, $\overline{f_\bullet}(W \wedge X) = 0$ implies $W \wedge X = 0$. $\square$

4.2. BA and DL.

Lemma 4.5. *When* $f_\bullet i_\bullet\langle X\rangle = \langle X\rangle$ *for all* $\langle X\rangle$*, the map* $i_\bullet(-)$ *sends* $\mathsf{BA}_{D(S)}$ *into* $\mathsf{BA}_{D(R)}$*. If* $\langle X\rangle \in \mathsf{BA}_{D(S)}$ *has complement* $\langle X^c\rangle$*, then* $\langle i_\bullet X\rangle \in \mathsf{BA}_{D(R)}$ *has complement* $\langle i_\bullet(X^c)\rangle \vee \langle M\rangle$*. In particular,* $\langle i_\bullet S\rangle$ *is complemented, with complement* $\langle M\rangle$ *(defined in 2.7).*

Proof. We first show that $\langle i_\bullet S\rangle$ is complemented, with complement $\langle M\rangle$. Note that $\langle i_\bullet S\rangle \equiv \langle R\rangle (\mathrm{mod}\ J)$, because $\overline{f_\bullet}\langle i_\bullet S\rangle = \langle f_\bullet i_\bullet S\rangle = \langle S\rangle = \langle f_\bullet R\rangle = \overline{f_\bullet}\langle R\rangle$, and $\overline{f_\bullet}$ is injective on BL/J. Therefore $\langle i_\bullet S\rangle \vee \langle M\rangle = \langle R\rangle \vee \langle M\rangle = \langle R\rangle$. On the other hand, $f_\bullet M = 0$, so $S \wedge f_\bullet M = 0$, so $i_\bullet S \wedge M = 0$.

We noted earlier that $\langle i_\bullet X\rangle \leq \langle i_\bullet S\rangle$ for all X. Therefore $i_\bullet X \wedge M = 0$ for all X.

Now suppose $\langle X\rangle \in \mathsf{BA}_{D(S)}$, so $\langle X\rangle \vee \langle X^c\rangle = \langle S\rangle$ and $\langle X\rangle \wedge \langle X^c\rangle = \langle 0\rangle$. This implies $\langle i_\bullet X\rangle \vee \langle i_\bullet X^c\rangle = \langle i_\bullet S\rangle$ and $\langle i_\bullet X\rangle \wedge \langle i_\bullet X^c\rangle = \langle 0\rangle$.

We calculate that

$$\langle i_\bullet X\rangle \vee (\langle i_\bullet X^c\rangle \vee \langle M\rangle) = \langle i_\bullet S\rangle \vee \langle M\rangle = \langle R\rangle \vee \langle M\rangle = \langle R\rangle,$$

because $\langle i_\bullet S\rangle \equiv \langle R\rangle (\mathrm{mod}\ J)$.

Also, we have

$$\begin{aligned}\langle i_\bullet X\rangle \wedge (\langle i_\bullet X^c\rangle \vee \langle M\rangle) &= (\langle i_\bullet X\rangle \wedge \langle i_\bullet X^c\rangle) \vee (\langle i_\bullet X\rangle \wedge \langle M\rangle)\\ &= \langle 0\rangle \vee (\langle i_\bullet X\rangle \wedge \langle M\rangle) = \langle 0\rangle.\end{aligned}$$

This shows that the complement of $\langle i_\bullet X\rangle$ is $\langle i_\bullet X^c\rangle \vee \langle M\rangle$. □

Proposition 4.6. *Suppose* $f_\bullet i_\bullet\langle X\rangle = \langle X\rangle$ *for all* $\langle X\rangle$*. The following hold.*

1. *The map* $f_\bullet$ *sends* $\mathsf{DL}_{D(R)}$ *onto* $\mathsf{DL}_{D(S)}$*, and the map* $i_\bullet$ *injects* $\mathsf{DL}_{D(S)}$ *into* $\mathsf{DL}_{D(R)}$*.*
2. *The map* $f_\bullet$ *sends* $\mathsf{BA}_{D(R)}$ *onto* $\mathsf{BA}_{D(S)}$*, and* $i_\bullet$ *injects* $\mathsf{BA}_{D(S)}$ *into* $\mathsf{BA}_{D(R)}$*.*
3. *The map* $\overline{f_\bullet}$ *establishes a poset isomorphism between (the image of)* DL *in* BL/J *and* DL *in* $D(S)$*, with inverse* $i_\bullet$*.*
4. *The map* $\overline{f_\bullet}$ *establishes a poset isomorphism between (the image of)* BA *in* BL/J *and* BA *in* $D(S)$*, with inverse* $i_\bullet$*.*

Proof. Lemma 4.1 implies that if $\langle Y\rangle = \langle Y \wedge Y\rangle$, then $\langle i_\bullet Y\rangle = \langle i_\bullet Y \wedge i_\bullet Y\rangle$, so $i_\bullet$ sends DL to DL. The rest follows from Propositions 2.9 and 4.2, Lemma 4.5, and the fact that $f_\bullet$ is surjective and $i_\bullet$ is injective. □

5. SURJECTIVE $f : R \to S$ WITH S NOETHERIAN, R NOT NECESSARILY NOETHERIAN

We know a lot about the Bousfield lattice of the derived category of a Noetherian ring S. In particular, $\mathsf{BL}_{D(S)}$ is isomorphic to the Boolean algebra on the classes $\{\langle k_{\mathfrak{p}}\rangle \mid \mathfrak{p} \in \mathsf{Spec}\, S\}$. Every localizing subcategory is a Bousfield class; every smashing localization is a finite localization; and there are classifications of localizing and thick subcategories, corresponding to subsets and specialization-closed subsets of $\mathsf{Spec}\, S$. Given $\mathfrak{p} \in \mathsf{Spec}\, S$, we have the finite Koszul object $S/\mathfrak{p}$, and

$K(\mathfrak{p}) := S_{\mathfrak{p}} \wedge S/\mathfrak{p}$. Furthermore, $\mathsf{loc}(K(\mathfrak{p})) = \mathsf{loc}(\overline{k_{\mathfrak{p}}})$. See [HPS97, Ch.6] or [Nee92] for details.

Much less is known in the general case of a commutative ring. In this section let $f : R \to S$ be surjective, with S Noetherian, but R not necessarily Noetherian. We will attempt to use $f_\bullet$ and $i_\bullet$ to pull back structure to $D(R)$.

Let $\mathfrak{p} \subseteq S$ be a prime ideal. Since S is Noetherian, choose generators $\mathfrak{p} = (z_1, ..., z_n)$. Since f is surjective, we can choose $y_i \in R$ such that $f(y_i) = z_i$ for all i. Define $\tilde{\mathfrak{p}} = (y_1, ..., y_n) \subseteq R$. Note that for every possible choice of y_i's, we'll always have $\tilde{\mathfrak{p}} \subseteq f^{-1}(\mathfrak{p})$. Also, $f(\tilde{\mathfrak{p}}) = \mathfrak{p}$.

As before, define the object $R/\tilde{\mathfrak{p}} = R/y_1 \wedge R/y_2 \wedge \cdots \wedge R/y_n$. Then define $K(\tilde{\mathfrak{p}}) := R/\tilde{\mathfrak{p}} \wedge R_{f^{-1}\mathfrak{p}}$.

It's clear that there is much choice involved in the definition of $K(\tilde{\mathfrak{p}})$. Let $\tilde{\mathfrak{p}}'$ be a different ideal of R, corresponding to a different choice of elements of $f^{-1}(z_i)$.

Question 5.1. *Do $R/\tilde{\mathfrak{p}}$ and $R/\tilde{\mathfrak{p}}'$ generate the same thick subcategory? Is $\langle K(\tilde{\mathfrak{p}})\rangle = \langle K(\tilde{\mathfrak{p}}')\rangle$?*

We'll see below that $\langle K(\tilde{\mathfrak{p}})\rangle$ is well-defined in BL/J, at least.

Lemma 5.2. *For all $\mathfrak{p} \in \mathsf{Spec}\, S$, we have $f_\bullet(R/\tilde{\mathfrak{p}}) = S/\mathfrak{p}$ and $f_\bullet K(\tilde{\mathfrak{p}}) = K(\mathfrak{p})$.*

Proof. For each i, if we apply $f_\bullet$ to the cofiber sequence $R \xrightarrow{y_i} R \longrightarrow R/y_i$, we get (see Remark 2.2)

$$\begin{aligned}&\left(f_\bullet R \xrightarrow{f_\bullet y_i} f_\bullet R \longrightarrow f_\bullet(R/y_i)\right)\\ &= \left(S \xrightarrow{f(y_i)} S \longrightarrow f_\bullet(R/y_i)\right)\\ &= \left(S \xrightarrow{z_i} S \longrightarrow f_\bullet(R/y_i)\right).\end{aligned}$$

Therefore $f_\bullet(R/y_i) = S/z_i$. Then using Lemma 3.3 we have

$$\begin{aligned}f_\bullet K(\tilde{\mathfrak{p}}) &= f_\bullet\left(R_{f^{-1}(\mathfrak{p})} \wedge R/y_1 \wedge \cdots \wedge R/y_n\right)\\ &= f_\bullet R_{f^{-1}(\mathfrak{p})} \wedge f_\bullet(R/y_1) \wedge \cdots \wedge f_\bullet(R/y_n)\\ &= S_{\mathfrak{p}} \wedge S/z_1 \wedge \cdots S/z_n = K(\mathfrak{p}).\end{aligned}$$

□

Lemma 5.3. *In $\mathsf{BL}_{D(R)}$ we have $\langle i_\bullet K(\mathfrak{p})\rangle \leq \langle K(\tilde{\mathfrak{p}})\rangle$, and in BL/J we have $\langle i_\bullet K(\mathfrak{p})\rangle = \langle K(\tilde{\mathfrak{p}})\rangle$. Therefore $\langle K(\tilde{\mathfrak{p}})\rangle$ is well-defined in BL/J, independent of choice of $\tilde{\mathfrak{p}}$.*

Proof. Suppose $X \wedge K(\tilde{\mathfrak{p}}) = 0$. Then $f_\bullet(X \wedge K(\tilde{\mathfrak{p}})) = f_\bullet X \wedge K(\mathfrak{p}) = 0$, so $X \wedge i_\bullet K(\mathfrak{p}) = 0$.

In BL/J, $\overline{f_\bullet}$ is injective, so we can follow the logic in the other way: if $X \wedge i_\bullet K(\mathfrak{p}) = 0$, then $f_\bullet X \wedge K(\mathfrak{p}) = 0$, so $\overline{f_\bullet}(X \wedge K(\tilde{\mathfrak{p}})) = 0$, so $X \wedge K(\tilde{\mathfrak{p}}) = 0$. □

Lemma 5.4. *For all X in $D(S)$, we have $f_\bullet i_\bullet \langle X\rangle = \langle X\rangle$.*

Proof. Every $\langle X\rangle$ in $\mathsf{BL}_{D(S)}$ is the join of some $\langle K(\mathfrak{p})\rangle$'s. Since $i_\bullet$ and $f_\bullet$ both preserve joins, we can reduce to the case where $\langle X\rangle = \langle K(\mathfrak{p})\rangle$. Lemma 5.3 showed $\langle i_\bullet K(\mathfrak{p})\rangle = \langle K(\tilde{\mathfrak{p}})\rangle$ in BL/J. Therefore $\overline{f_\bullet}\langle i_\bullet K(\mathfrak{p})\rangle = \overline{f_\bullet}\langle K(\tilde{\mathfrak{p}})\rangle$, so $\langle f_\bullet i_\bullet K(\mathfrak{p})\rangle = \langle K(\mathfrak{p})\rangle$. □

Therefore the results of Section 4 hold, and we have a poset bijection $\overline{f_\bullet} : \mathsf{BL}/J \to \mathsf{BL}_{D(S)}$, with inverse $i_\bullet(-)$. This immediately gives the following

Proposition 5.5. *In* BL/J *the following hold.*

1. *If* $\mathfrak{p} \neq \mathfrak{p}'$, *then* $\langle K(\tilde{\mathfrak{p}})\rangle \wedge \langle K(\tilde{\mathfrak{p}}')\rangle = \langle 0\rangle$.
2. $\langle K(\tilde{\mathfrak{p}})\rangle \wedge \langle K(\tilde{\mathfrak{p}})\rangle = \langle K(\tilde{\mathfrak{p}})\rangle$.
3. $\langle K(\tilde{\mathfrak{p}})\rangle$ *is a minimal nonzero Bousfield class.*
4. $\langle R\rangle = \coprod_{p\in \mathsf{Spec}\, S} \langle K(\tilde{\mathfrak{p}})\rangle$.

Proof. All the proofs work by pushing into $\mathsf{DL}_{D(S)}$. For example, we'll prove (4). Suppose X in $D(R)$ has $X \wedge K(\tilde{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \mathsf{Spec}\, S$. We want to show that $X = 0$. But we know $f_\bullet X \wedge f_\bullet K(\tilde{\mathfrak{p}}) = 0$, which is to say $f_\bullet X \wedge K(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathsf{Spec}\, S$. Since $\langle S\rangle = \coprod_{\mathsf{Spec}\, S}\langle K(\mathfrak{p})\rangle$, this means $f_\bullet X = 0$. Since $f_\bullet$ is injective on BL/J, we have $X = 0$. □

Question 5.6. *To what extent can these results be pulled back to* $\mathsf{BL}_{D(R)}$*? For example, (1) might hold in* $\mathsf{BL}_{D(R)}$*, but (4) most definitely does not. It would be interesting to prove or disprove (3) in* $\mathsf{BL}_{D(R)}$*. The skew field objects would probably be useful here.*

The following lemmas will be used later.

Lemma 5.7. *Let* $\tilde{\mathfrak{p}} = (y_1, ..., y_n)$ *be as above. Then*

$$\langle R/\tilde{\mathfrak{p}}\rangle \leq \cdots \leq \langle R/(y_1, y_2)\rangle \leq \langle R/y_1\rangle.$$

Proof. This comes from repeated application of Lemma 2.5. For example, $(y_1) \subseteq (y_1, y_2)$ implies that $R/(y_1, y_2) \in \mathsf{th}(R/y_1)$, so $\langle R/(y_1, y_2)\rangle \leq \langle R/y_1\rangle$. □

Lemma 5.8. *If* $\mathfrak{p} \nsubseteq \mathfrak{p}'$ *are prime ideals of* S, *then*

$$R/\tilde{\mathfrak{p}} \wedge R_{f^{-1}(\mathfrak{p}')} = 0 \text{ and } R/\tilde{\mathfrak{p}} \wedge \overline{k_{f^{-1}(\mathfrak{p}')}} = 0.$$

Proof. Take $z_1 \in \mathfrak{p}\backslash\mathfrak{p}'$, and complete it with z_i so that $(z_1, z_2, ..., z_n) = \mathfrak{p}$. For each i, choose $y_i \in f^{-1}(z_i)$, and define $\tilde{\mathfrak{p}} = (y_1, ..., y_n)$. Note that $y_1 \in f^{-1}(\mathfrak{p})$ but $y_1 \notin f^{-1}(\mathfrak{p}')$.

Now, in $D(R)$, the object R/y_1 is represented by the chain complex

$$\cdots \to 0 \to R \xrightarrow{y_1} R \to 0 \to \cdots.$$

Therefore $R/y_1 \wedge R_{f^{-1}(\mathfrak{p}')}$ is the chain complex

$$\cdots \to 0 \to R_{f^{-1}(\mathfrak{p}')} \xrightarrow{y_1} R_{f^{-1}(\mathfrak{p}')} \to 0 \to \cdots.$$

Since $y_1 \in R\backslash f^{-1}(\mathfrak{p}')$, it is invertible in $R_{f^{-1}(\mathfrak{p}')}$. Therefore the above complex is acyclic, and $R/y_1 \wedge R_{f^{-1}(\mathfrak{p}')} = 0$.

The previous lemma tells us that $\langle R/\tilde{\mathfrak{p}}\rangle \leq \langle R/y_1\rangle$, so $R/\tilde{\mathfrak{p}} \wedge R_{f^{-1}(\mathfrak{p}')} = 0$.

Similarly, $R/y_1 \wedge \overline{k_{f^{-1}(\mathfrak{p}')}}$ is the chain complex

$$\cdots \to 0 \to \frac{R_{f^{-1}(\mathfrak{p}')}}{f^{-1}(\mathfrak{p}')R_{f^{-1}(\mathfrak{p}')}} \xrightarrow{y_1} \frac{R_{f^{-1}(\mathfrak{p}')}}{f^{-1}(\mathfrak{p}')R_{f^{-1}(\mathfrak{p}')}} \to 0 \to \cdots.$$

The map y_1 is also invertible here, so this chain complex is acyclic, and we conclude $R/\tilde{\mathfrak{p}} \wedge \overline{k_{f^{-1}(\mathfrak{p}')}} = 0$. □

5.1. Thick subcategories and support.

Thomason [Tho97] gave a way of getting at thick subcategories of finite objects in $D(R)$, via subsets of $\mathsf{Spec}\, R$. Here we show how $f_\bullet$ and $i_\bullet$ relate the thick subcategories of $D(R)$ and $D(S)$, and how they work with $f^{-1} : \mathsf{Spec}\, S \to \mathsf{Spec}\, R$.

In the next section, where we assume R is Noetherian, we will improve on these results and extend them to localizing subcategory classification. With this in mind, we give the following definitions.

Definition 5.9. *Given a thick subcategory $\mathcal{A}$ of finite objects in $D(R)$, define $f_\bullet\mathcal{A}$ to be the intersection of all thick subcategories of finite objects in $D(S)$ that contain $f_\bullet X$ for all X in $\mathcal{A}$. Given localizing subcategories $\mathcal{C}$ of $D(R)$ and $\mathcal{D}$ of $D(S)$, define $f_\bullet\mathcal{C}$ and $i_\bullet\mathcal{D}$ similarly.*

We know that $f_\bullet$ takes finite objects to finite objects, but in general $i_\bullet$ does not. Thus we do not have a nice operation of $i_\bullet$ on thick subcategories of finite objects of $D(S)$. See Section 5.2 however.

Note that for principal subcategories we have $f_\bullet(\mathsf{th}(X)) = \mathsf{th}(f_\bullet X)$ and $f_\bullet(\mathsf{loc}(X)) = \mathsf{loc}(f_\bullet X)$, and likewise for $i_\bullet$. Also, since the full subcategory of finite objects is essentially small in the derived category of any commutative ring, if $\mathcal{A}$ is a thick subcategory of finite objects, we can identify $f_\bullet\mathcal{A}$ with the thick subcategory generated by the set $\{f_\bullet X \ : \ X \in \mathcal{A}\}$.

For the Noetherian ring S, the classification gives a one-to-one correspondence between thick subcategories of finite objects in $D(S)$ and specialization-closed subsets of $\mathsf{Spec}\, S$, via the notion of support. For an object X in $D(S)$, $\mathrm{supp}(X) = \{\mathfrak{p} \in \mathsf{Spec}\, S \ : \ \overline{k_\mathfrak{p}} \wedge X \neq 0\}$. A specialization-closed subset $W \subseteq \mathsf{Spec}\, S$ corresponds to

$$\mathsf{th}\,(S/\mathfrak{p} \ : \ \mathfrak{p} \in W) = \{X \text{ in } \mathcal{F} \ : \ \mathrm{supp}(X) \subseteq W\},$$

and a thick subcategory $\mathcal{A}$ corresponds to

$$\mathrm{supp}(\mathcal{A}) = \{\mathfrak{p} \in \mathsf{Spec}\, S \ : \ \text{ there exists } X \text{ in } \mathcal{A} \text{ with } \mathfrak{p} \in \mathrm{supp}(X)\}.$$

For a commutative ring R that is not necessarily Noetherian, we replace specialization-closed subsets with *Thomason-closed* subsets of $\mathsf{Spec}\, R$, which are those of the form $\bigcup_\alpha V(I_\alpha)$, where each I_α is finitely generated.

Question 5.10. *Does the function $f^{-1} : \mathsf{Spec}\, S \to \mathsf{Spec}\, R$ take specialization-closed subsets to Thomason-closed subsets? (See Remark 5.16)*

Throughout this section, fix $T := f^{-1}(\mathsf{Spec}\, S) \subseteq \mathsf{Spec}\, R$, and $U := \mathsf{Spec}\, R \backslash T$. Our computations rely on the following observation.

Lemma 5.11. *Given $\mathfrak{p} \in \mathsf{Spec}\, R$, we have*

$$f_\bullet \overline{k_\mathfrak{q}} = \begin{cases} \overline{k_{f(\mathfrak{q})}}, & \text{if } \mathfrak{q} \in T \\ 0, & \text{if } \mathfrak{q} \in U \end{cases}$$

Proof. In Proposition 3.4 we showed that $f_\bullet(\overline{k_{f^{-1}\mathfrak{p}}}) = \overline{k_\mathfrak{p}}$. Now take $\mathfrak{q} \in U$. Then for all $\mathfrak{p} \in \mathsf{Spec}\, S$, $f^{-1}\mathfrak{p} \neq \mathfrak{q}$. By Lemma 2.5, $\overline{k_{f^{-1}\mathfrak{p}}} \wedge \overline{k_\mathfrak{q}} = 0$. Therefore

$$0 = f_\bullet(0) = f_\bullet(\overline{k_{f^{-1}\mathfrak{p}}} \wedge \overline{k_\mathfrak{q}}) = f_\bullet\overline{k_{f^{-1}\mathfrak{p}}} \wedge f_\bullet\overline{k_\mathfrak{q}} = \overline{k_\mathfrak{p}} \wedge f_\bullet\overline{k_\mathfrak{q}},$$

for all $\mathfrak{p} \in \mathsf{Spec}\, S$. Since S is Noetherian, we have

$$\langle S \rangle = \coprod_{\mathfrak{p} \in \mathsf{Spec}\, S} \langle \overline{k_\mathfrak{p}} \rangle.$$

This implies that $f_\bullet \overline{k_\mathfrak{q}} = f_\bullet \overline{k_\mathfrak{q}} \wedge S = 0$. ☐

Proposition 5.12. *Let X be an arbitrary element of $D(S)$. Then we have*

$$\mathit{supp}(i_\bullet X) = f^{-1}(\mathit{supp}(X)).$$

Proof. A prime $\mathfrak{q}$ is in $\operatorname{supp}(i_\bullet X)$ if and only if $\overline{k_\mathfrak{q}} \wedge i_\bullet X \neq 0$. The projection formula says $\overline{k_\mathfrak{q}} \wedge i_\bullet X = i_\bullet(f_\bullet \overline{k_\mathfrak{q}} \wedge X)$, and $i_\bullet$ is injective, so $\mathfrak{q}$ is in $\operatorname{supp}(i_\bullet X)$ if and only if $f_\bullet \overline{k_\mathfrak{q}} \wedge X \neq 0$.

If $f_\bullet \overline{k_\mathfrak{q}} \wedge X \neq 0$ then $f_\bullet \overline{k_\mathfrak{q}} \neq 0$, so Lemma 5.11 forces $\mathfrak{q} \in f^{-1}(\mathsf{Spec}\, S)$, say $\mathfrak{q} = f^{-1}\mathfrak{p}$. Then by Proposition 3.4, $f_\bullet \overline{k_\mathfrak{q}} \wedge X = \overline{k_\mathfrak{p}} \wedge X$, so $\mathfrak{p} \in \operatorname{supp} X$, and $\mathfrak{q} = f^{-1}\mathfrak{p} \in f^{-1}(\operatorname{supp} X)$.

If $\mathfrak{q} = f^{-1}\mathfrak{p} \in f^{-1}(\operatorname{supp} X)$ then by Proposition 3.4, $0 \neq \overline{k_\mathfrak{p}} \wedge X = f_\bullet \overline{k_\mathfrak{q}} \wedge X$. ☐

Corollary 5.13. *Take $\mathfrak{p} \in \mathsf{Spec}\, S$. Then*

$$\mathit{supp}(i(S/\mathfrak{p})) = f^{-1}(V(\mathfrak{p})).$$

Proof. Proposition 6.1.7(c) in [HPS97] shows that $\operatorname{supp}(S/\mathfrak{p}) = V(\mathfrak{p})$. ☐

Lemma 5.14. *Take $\mathfrak{p} \in \mathsf{Spec}\, S$. Then*

$$f^{-1}(V(\mathfrak{p})) = V(f^{-1}(\mathfrak{p})).$$

Proof. Given $\mathfrak{q} \in f^{-1}(V(\mathfrak{p}))$, we have $\mathfrak{q} = f^{-1}(\mathfrak{r})$ for some $\mathfrak{r}$ with $\mathfrak{p} \subseteq \mathfrak{r}$. Then $f^{-1}(\mathfrak{p}) \subseteq f^{-1}(\mathfrak{r}) = \mathfrak{q}$. Thus $\mathfrak{q} \in V(f^{-1}(\mathfrak{p}))$.

It's clear that $f^{-1}\mathfrak{p} \in f^{-1}(V(\mathfrak{p}))$. Since $V(\mathfrak{p})$ is a closed subset of $\mathsf{Spec}\, S$, and f^{-1} is a homeomorphism onto its image in $\mathsf{Spec}\, R$ (see Lemma 3.2), then $f^{-1}(V(\mathfrak{p}))$ is a closed subset of $\mathsf{Spec}\, R$. This implies that the closure $V(f^{-1}(\mathfrak{p}))$ is contained in $f^{-1}(V(\mathfrak{p}))$. ☐

Lemma 5.15. *Take $\mathfrak{p} \in \mathsf{Spec}\, S$. Then*

$$f^{-1}(V(\mathfrak{p})) = V(\tilde{\mathfrak{p}}) \cap T.$$

Proof. Take $\mathfrak{q} \in V(\tilde{\mathfrak{p}}) \cap T$, and take $\mathfrak{r} \in \mathsf{Spec}\, S$ so $f^{-1}(\mathfrak{r}) = \mathfrak{q}$. Then $\tilde{\mathfrak{p}} \subseteq \mathfrak{q}$, so $\mathfrak{p} = f(\tilde{\mathfrak{p}}) \subseteq f(\mathfrak{q}) = \mathfrak{r}$. Thus $\mathfrak{r} \in V(\mathfrak{p})$, so $\mathfrak{q} \in f^{-1}(V(\mathfrak{p}))$.

Since $f^{-1}(V(\mathfrak{p})) \subseteq f^{-1}(\mathsf{Spec}\, S)$, it remains to show that $f^{-1}(V(\mathfrak{p})) \subseteq V(\tilde{\mathfrak{p}})$. In light of Corollary 5.13, take $\mathfrak{q} \in \operatorname{supp}(i(S/\mathfrak{p}))$, so $i(S/\mathfrak{p}) \wedge \overline{k_\mathfrak{q}} \neq 0$. Suppose $\mathfrak{p} = (z_1, ..., z_n)$. We will show that every choice of $f^{-1}(z_i)$ is in $\mathfrak{q}$ for all i; this will imply that $\tilde{\mathfrak{p}} \subseteq \mathfrak{q}$, so $\mathfrak{q} \in V(\tilde{\mathfrak{p}})$.

Suppose, towards a contradiction, that there is some $f^{-1}(z_i)$ not contained in $\mathfrak{q}$. As a map of R-modules,

$$R_\mathfrak{q} \xrightarrow{f^{-1}(z_i)} R_\mathfrak{q}$$

is an isomorphism; as a map of objects in $D(R)$ it is an equivalence. Therefore applying $f_\bullet$,

$$f_\bullet R_\mathfrak{q} \xrightarrow{z_i} f_\bullet R_\mathfrak{q}$$

is an equivalence. Applying $- \wedge f_\bullet(R/\mathfrak{q}R)$ and noting that $f_\bullet R_\mathfrak{q} \wedge f_\bullet(R/\mathfrak{q}R) = f_\bullet \overline{k_\mathfrak{q}}$, we see that $f_\bullet \overline{k_\mathfrak{q}} \xrightarrow{z_i} f_\bullet \overline{k_\mathfrak{q}}$ is an equivalence. The cofiber $S/z_i \wedge f_\bullet \overline{k_\mathfrak{q}}$ is zero. Smashing with the other S/z_j, we see that $S/\mathfrak{p} \wedge f_\bullet \overline{k_\mathfrak{q}} = 0$. But $i_\bullet(S/\mathfrak{p} \wedge f_\bullet \overline{k_\mathfrak{q}}) = i(S/\mathfrak{p}) \wedge \overline{k_\mathfrak{q}} \neq 0$ by hypothesis. ☐

Remark 5.16. Given a specialization-closed subset W of $\mathsf{Spec}\,S$, it is not hard to see that

$$f^{-1}(W) = \bigcup_{\mathfrak{p}\in W} f^{-1}(V(\mathfrak{p})).$$

Combining this with the last result gives

$$f^{-1}(W) = \left(\bigcup_{\mathfrak{p}\in W} V(\tilde{\mathfrak{p}})\right) \cap T.$$

In a Noetherian ring S, the Koszul objects $S/\mathfrak{p}$ have $\mathrm{supp}(S/\mathfrak{p}) = V(\mathfrak{p})$. The next two lemmas give information about $\mathrm{supp}(R/\tilde{\mathfrak{p}})$ in the general case.

Lemma 5.17. *Take* $\mathfrak{p} \in \mathsf{Spec}\,S$ *and consider* $R/\tilde{\mathfrak{p}}$ *in* $D(R)$*. Then*

$$\mathit{supp}(R/\tilde{\mathfrak{p}}) \subseteq V(\tilde{\mathfrak{p}}).$$

Proof. Suppose $\mathfrak{q} \in \mathsf{Spec}\,R$ has $R/\tilde{\mathfrak{p}} \wedge \overline{k_{\mathfrak{q}}} \neq 0$. We must show that $\tilde{\mathfrak{p}} \subseteq \mathfrak{q}$.

Suppose $\mathfrak{p} = (z_1, ..., z_n)$, and let $y_i \in f^{-1}(z_i)$, for $1 \leq i \leq n$, be choices of preimages, so that $\tilde{\mathfrak{p}} = (y_1, ..., y_n)$. Then for each i, Lemma 5.8 implies that $R/y_i \wedge \overline{k_{\mathfrak{q}}} \neq 0$.

If we consider the triangle

$$R \wedge \overline{k_{\mathfrak{q}}} \xrightarrow{y_i \wedge 1} R \wedge \overline{k_{\mathfrak{q}}} \longrightarrow R/y_i \wedge \overline{k_{\mathfrak{q}}},$$

we see that the map

$$\overline{k_{\mathfrak{q}}} \xrightarrow{y_i} \overline{k_{\mathfrak{q}}}$$

is not an equivalence. Thus as a map of R-modules, $y_i : k_{\mathfrak{q}} \to k_{\mathfrak{q}}$ is not an isomorphism. But $y_i \in R$, and $k_{\mathfrak{q}}$ is $R_{\mathfrak{q}}$-local, so everything in $R\backslash\mathfrak{q}$ is invertible. Therefore $y_i \in \mathfrak{q}$. Since this is true for all i, $\tilde{\mathfrak{p}} \subseteq \mathfrak{q}$. □

Lemma 5.18. *Given* $\mathfrak{p} \in \mathsf{Spec}\,S$*, we have*

$$\mathit{supp}(R/\tilde{\mathfrak{p}}) \cap T = f^{-1}(V(\mathfrak{p})).$$

So although we haven't shown whether the containment $\mathrm{supp}(R/\tilde{\mathfrak{p}}) \subseteq V(\tilde{\mathfrak{p}})$ is an equality, we know their intersections with $f^{-1}(\mathsf{Spec}\,S)$ are equal.

Proof. The previous two lemmas imply that $\mathrm{supp}(R/\tilde{\mathfrak{p}}) \cap T \subseteq V(\tilde{\mathfrak{p}}) \cap T = f^{-1}(V(\mathfrak{p}))$. This also follows from the contrapositive of Lemma 5.8.

Now take $\mathfrak{q} \in f^{-1}(V(\mathfrak{p}))$. Clearly $\mathfrak{q} \in f^{-1}(\mathsf{Spec}\,S)$; we must show that $R/\tilde{\mathfrak{p}} \wedge \overline{k_{\mathfrak{q}}} \neq 0$. There is some $\mathfrak{r} \in \mathsf{Spec}\,S$ such that $\mathfrak{q} = f^{-1}(\mathfrak{r})$. Then $f^{-1}(\mathfrak{r}) \in f^{-1}(V(\mathfrak{p}))$ implies that $\mathfrak{r} \in V(\mathfrak{p}) = \mathrm{supp}(S/\mathfrak{p})$, so $S/\mathfrak{p} \wedge \overline{k_{\mathfrak{r}}} \neq 0$. The computation

$$f_\bullet(R/\tilde{\mathfrak{p}} \wedge \overline{k_{\mathfrak{q}}}) = f_\bullet(R/\tilde{\mathfrak{p}}) \wedge f_\bullet(\overline{k_{f^{-1}(\mathfrak{r})}}) = S/\mathfrak{p} \wedge \overline{k_{\mathfrak{r}}} \neq 0$$

shows that $R/\tilde{\mathfrak{p}} \wedge \overline{k_{\mathfrak{q}}} \neq 0$. □

Question 5.19. *Is* $\mathit{supp}(R/\tilde{\mathfrak{p}}) = V(\tilde{\mathfrak{p}})$*?*

Remark 5.20. If $W \subseteq \mathsf{Spec}\,S$ is specialization-closed, then combining the last result with Remark 5.16 and Corollary 5.13 gives

$$\begin{aligned} f^{-1}(W) &= \mathrm{supp}\left(\mathsf{th}\left(R/\tilde{\mathfrak{p}} \ : \ \mathfrak{p} \in W\right)\right) \cap T \\ &= \mathrm{supp}\left(\mathsf{th}(i_\bullet(S/\mathfrak{p})) \ : \ \mathfrak{p} \in W\right). \end{aligned}$$

Now we will look at how $f_\bullet$ works with thick subcategories and the classification. This diagram is helpful to keep in mind.

$$\begin{tikzcd}
\left\{\begin{matrix}\text{nice subsets}\\ \text{of } \mathsf{Spec}\, R\end{matrix}\right\} \arrow[d, shift left] & & \left\{\begin{matrix}\text{nice subsets}\\ \text{of } \mathsf{Spec}\, S\end{matrix}\right\} \arrow[ll, "f^{-1}"'] \arrow[d, shift left] \\
\left\{\begin{matrix}\text{thick subcategories of}\\ \text{finite objects in } D(R)\end{matrix}\right\} \arrow[u, shift left, "supp(-)"] \arrow[rr, "f_\bullet"] & & \left\{\begin{matrix}\text{thick subcategories of}\\ \text{finite objects in } D(S)\end{matrix}\right\} \arrow[u, shift left, "supp(-)"]
\end{tikzcd}$$

Proposition 5.21. *Let $\mathcal{B}$ be a thick subcategory of finite objects in $D(R)$, and let X be an arbitrary element of $D(R)$. Then*

$$f^{-1}(supp(f_\bullet X)) \subseteq supp(X) \cap T, \text{ and } f^{-1}(supp(f_\bullet \mathcal{B})) \subseteq supp(\mathcal{B}) \cap T.$$

Equality holds when $X = R/\tilde{\mathfrak{p}}$, or when $X = i_\bullet Y$ for some $Y \in D(S)$.

Proof. Given $\mathfrak{q} = f^{-1}(\mathfrak{r}) \in f^{-1}(\text{supp}(f_\bullet X))$,

$$0 \neq f_\bullet X \wedge \overline{k_{\mathfrak{r}}} = f_\bullet(X \wedge \overline{k_{f^{-1}(\mathfrak{r})}}),$$

so $X \wedge \overline{k_{f^{-1}(\mathfrak{r})}} \neq 0$. Therefore $f^{-1}(\mathfrak{r}) = \mathfrak{q} \in \text{supp}(X) \cap T$.

Because $f_\bullet \mathcal{B}$ is the thick subcategory generated by all the $f_\bullet X$, $X \in \mathcal{B}$, we have that

$$Y \wedge f_\bullet X = 0 \text{ for all } X \in \mathcal{B} \text{ if and only if } Y \wedge W = 0 \text{ for all } W \in f_\bullet \mathcal{B}.$$

Therefore, given $\mathfrak{q} =\in f^{-1}(\text{supp}(f_\bullet \mathcal{B}))$, there is some $X \in \mathcal{B}$ with

$$\mathfrak{q} \in f^{-1}(\text{supp}(f_\bullet X)) \subseteq \text{supp}(X) \cap T \subseteq \text{supp}(\mathcal{B}) \cap T.$$

When $X = R/\tilde{\mathfrak{p}}$, we have

$$f^{-1}(\text{supp}(f_\bullet(R/\tilde{\mathfrak{p}}))) = f^{-1}(\text{supp}(S/\mathfrak{p})) = f^{-1}(V(\mathfrak{p})) = \text{supp}(R/\tilde{\mathfrak{p}}) \cap T.$$

Now suppose $X = i_\bullet Y$ for some $Y \in D(S)$, and take $\mathfrak{q} = f^{-1}(\mathfrak{r}) \in \text{supp}(X) \cap T$. Then $i_\bullet Y \wedge \overline{k_{f^{-1}(\mathfrak{r})}} \neq 0$, and the projection formula implies that $0 \neq Y \wedge f_\bullet \overline{k_{f^{-1}(\mathfrak{r})}} = Y \wedge \overline{k_{\mathfrak{r}}}$. By the injectivity of $i_\bullet$, and Lemma 4.1, this implies $i_\bullet Y \wedge i_\bullet \overline{k_{\mathfrak{r}}} \neq 0$. Again, the projection formula gives $i_\bullet(f_\bullet X \wedge \overline{k_{\mathfrak{r}}}) \neq 0$, so $f_\bullet X \wedge \overline{k_{\mathfrak{r}}} \neq 0$, and $\mathfrak{q} = f^{-1}(\mathfrak{r}) \in f^{-1}(\text{supp}(f_\bullet X))$. □

5.2. **When $i_\bullet S$ is finite.** If we assume that $i_\bullet S$ is a finite object in $D(R)$, then $i_\bullet$ sends finite objects to finite objects, and gives a well-defined operation on thick subcategories of finite objects in $D(S)$.

Furthermore, the objects $i_\bullet(S/\mathfrak{p})$ will be finite, for all $\mathfrak{p} \in \mathsf{Spec}\, S$. Remark 5.20 implies that $f^{-1}(W)$ will be a Thomason-closed subset when $W \subseteq \mathsf{Spec}\, S$ is a specialization-closed subset. This also follows from the next proposition.

Proposition 5.22. *Suppose $i_\bullet S$ is finite, and let $\mathcal{A}$ be a thick subcategory of finite objects in $D(S)$. Then*

$$supp(i_\bullet \mathcal{A}) = f^{-1}(supp(\mathcal{A})).$$

Proof. Because $i_\bullet \mathcal{A}$ is the thick subcategory generated by all the $i_\bullet X$, $X \in \mathcal{A}$, we have that

$$Y \wedge i_\bullet X = 0 \text{ for all } X \in \mathcal{A} \text{ if and only if } Y \wedge W = 0 \text{ for all } W \in i_\bullet \mathcal{A}.$$

This implies that, for $\mathfrak{q} \in \mathsf{Spec}\, R$, $\overline{k_\mathfrak{q}} \wedge W \neq 0$ for some $W \in i_\bullet \mathcal{A}$ if and only if $\overline{k_\mathfrak{q}} \wedge i_\bullet X \neq 0$ for some $X \in \mathcal{A}$. Using this, the argument in Proposition 5.12 gives that $\operatorname{supp}(i_\bullet \mathcal{A}) = f^{-1}(\operatorname{supp}(\mathcal{A}))$. $\square$

Lemma 5.23. *Assume $i_\bullet S$ is finite. For any finite object X in $D(S)$, we have*

$$\mathsf{th}(f_\bullet i_\bullet X) = \mathsf{th}(X).$$

Proof. Because X is finite and S is Noetherian, $\mathsf{th}(X) = \mathsf{th}(S/\mathfrak{p} \ : \ \mathfrak{p} \in W)$ for some $W \subseteq \mathsf{Spec}\, S$. Specifically, $W = \operatorname{supp}(X)$. Since $f_\bullet$ and $i_\bullet$ are exact and commute with coproducts, we get

$$\mathsf{th}(f_\bullet i_\bullet X) = \mathsf{th}\left(f_\bullet i_\bullet (S/\mathfrak{p}) \ : \ \mathfrak{p} \in W\right).$$

Thus we can reduce to showing that $\mathsf{th}(f_\bullet i_\bullet (S/\mathfrak{p})) = \mathsf{th}(S/\mathfrak{p})$. Since $f_\bullet i_\bullet S$ is finite, these are both thick subcategories of finite objects, so if suffices to show that they have the same support.

Proposition 5.21 says that

$$f^{-1}(\operatorname{supp}(f_\bullet i_\bullet (S/\mathfrak{p}))) = \operatorname{supp}(i_\bullet (S/\mathfrak{p})) \cap T = f^{-1}(V(\mathfrak{p})).$$

Therefore $\operatorname{supp}(f_\bullet i_\bullet (S/\mathfrak{p})) = V(\mathfrak{p}) = \operatorname{supp}(S/\mathfrak{p})$. $\square$

Question 5.24. *Is there a way to pull these two results to get equality in Proposition 5.21, when $i_\bullet S$ is finite?*

5.3. An example.

Fix integers $n_i > 1$ and a prime p, and define

$$\Gamma := \frac{\mathbb{Z}_{(p)}[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)} \text{ and } \Lambda := \frac{\mathbb{F}_p[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}.$$

The Bousfield lattice of $D(\Lambda)$ is studied extensively in [DP08]. Consider the projection map $f : \Gamma \to \mathbb{Z}_{(p)}$. Then $\mathsf{Spec}\, \mathbb{Z}_{(p)} = \{(0), (p)\}$, and

$$\mathsf{Spec}\, \Gamma = \{(x_1, x_2, ...), (p, x_1, x_2, ...)\}.$$

The map $f^{-1} : \mathsf{Spec}\, \mathbb{Z}_{(p)} \to \mathsf{Spec}\, \Gamma$ is a bijection. Choosing $p \in f^{-1}(p)$, the prime ideal $\mathfrak{p} = (p) \subseteq \mathbb{Z}_{(p)}$ pulls back to $\tilde{\mathfrak{p}} = (p) \subseteq \Gamma$, and we have $V(\tilde{\mathfrak{p}}) = \{(p, x_1, x_2, ...)\}$. Thus there are three Thomason-closed subsets of $\mathsf{Spec}\, \Gamma$:

$$\emptyset, \{(p, x_1, x_2, ...)\}, \text{ and } \mathsf{Spec}\, \Gamma.$$

Let $\mathcal{D} = \{W \in \mathcal{F} \ : \ \operatorname{supp}(W) \subseteq \{(p, x_1, x_2, ...)\}\}$ be the thick subcategory of finite objects corresponding to the unique proper nonempty Thomason-closed subset of $\mathsf{Spec}\, \Gamma$.

The object $\Gamma/\tilde{\mathfrak{p}}$ in $D(\Gamma)$ is the cofiber of $\Gamma \xrightarrow{p} \Gamma$, so is equivalent to the chain complex

$$\left(\cdots \to 0 \to \Gamma \xrightarrow{p} \Gamma \to 0 \to \cdots\right).$$

Lemma 5.17 shows that $\operatorname{supp}(\Gamma/\tilde{\mathfrak{p}}) \subseteq V(\tilde{\mathfrak{p}}) = \{(p, x_1, x_2, ...)\}$, so $\Gamma/\tilde{\mathfrak{p}} \in \mathcal{D}$. Therefore $\mathsf{th}(\Gamma/\tilde{\mathfrak{p}}) \subseteq \mathcal{D}$, and so from the classification we see that

$$\mathsf{th}(\Gamma/\tilde{\mathfrak{p}}) = \mathsf{th}\left(\cdots \to 0 \to \Gamma \xrightarrow{p} \Gamma \to 0 \to \cdots\right) = \mathcal{D}.$$

In Section 7 we will connect this example to the Bousfield lattice of $D(\Lambda)$.

6. Surjective $f : R \to S$, with R and S both Noetherian

All the results from the last section can be improved and generalized when R is also Noetherian. In Section 6.3 we show that the map $\overline{f_\bullet}$ induces a splitting

$$\mathsf{BL}_{D(R)} \cong \mathsf{BL}_{D(S)} \times J.$$

6.1. **Localizing subcategories.** The localizing subcategories of $D(R)$ and $D(S)$ are classified by arbitrary subsets of $\mathsf{Spec}\, R$ and $\mathsf{Spec}\, S$. More specifically, the lattice $\mathbf{2}^{\mathsf{Spec}\, R}$ of subsets of $\mathsf{Spec}\, R$ is isomorphic to the lattice of localizing subcategories of $D(R)$, denoted $\mathsf{Loc}(D(R))$. Under this isomorphism, $\{\mathfrak{p}\}$ corresponds to $\mathsf{loc}(\overline{k_{\mathfrak{p}}})$; in particular, $\mathrm{supp}(\overline{k_{\mathfrak{p}}}) = \{\mathfrak{p}\}$.

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{arbitrary subsets}\\ \text{of } \mathsf{Spec}\, R\end{array}\right\} \arrow[r, shift right, "(f^{-1})^{-1}"'] \arrow[d, shift right, "supp(-)"'] & \left\{\begin{array}{c}\text{arbitrary subsets}\\ \text{of } \mathsf{Spec}\, S\end{array}\right\} \arrow[l, shift right, "f^{-1}"'] \arrow[d, shift right, "supp(-)"'] \\
\left\{\begin{array}{c}\text{localizing subcategories}\\ \text{in } D(R)\end{array}\right\} \arrow[u, shift right] \arrow[r, shift left, "f_\bullet"] & \left\{\begin{array}{c}\text{localizing subcategories}\\ \text{in } D(S)\end{array}\right\} \arrow[u, shift right] \arrow[l, shift left, "i_\bullet"]
\end{tikzcd}$$

Throughout this section, fix $T := f^{-1}(\mathsf{Spec}\, S) \subseteq \mathsf{Spec}\, R$, and $U := \mathsf{Spec}\, R \backslash T$.

The map f^{-1} induces a map $\mathbf{2}^{\mathsf{Spec}\, S} \to \mathbf{2}^{\mathsf{Spec}\, R}$, and $f_\bullet$ induces a map $\mathsf{Loc}(D(R)) \to \mathsf{Loc}(D(S))$. The following lemma states that, via the lattice isomorphisms described above, we have "$(f^{-1})^{-1} = f_\bullet$".

Proposition 6.1. *Let $\mathcal{A}$ be a localizing subcategory of $D(R)$. Then*

$$supp(f_\bullet \mathcal{A}) = (f^{-1})^{-1} \left(supp(\mathcal{A})\right) = f \left(supp(\mathcal{A}) \cap T\right).$$

Equivalently, $f^{-1} \left(supp(f_\bullet \mathcal{A})\right) = supp(\mathcal{A}) \cap T$. The same holds if we replace $\mathcal{A}$ with X, for any object X in $D(R)$.

Proof. Let $\mathcal{A}$ be a localizing subcategory of $D(R)$. Then $\mathcal{A} = \mathsf{loc}(\overline{k_{\mathsf{q}}} \;:\; \mathsf{q} \in W)$ for some $W \subseteq \mathsf{Spec}\, R$, and in this case $\mathrm{supp}(\mathcal{A}) = W$. Using Lemma 5.11, we compute that

$$f_\bullet \mathcal{A} = f_\bullet \mathsf{loc}\left(\overline{k_{\mathsf{q}}} \;:\; \mathsf{q} \in W\right) = \mathsf{loc}\left(f_\bullet \overline{k_{\mathsf{q}}} \;:\; \mathsf{q} \in W\right) = \mathsf{loc}\left(\overline{k_{f(\mathsf{q})}} \;:\; \mathsf{q} \in W \cap T\right).$$

In other words, $\mathrm{supp}(f_\bullet \mathcal{A}) = f(W \cap T) = f(\mathrm{supp}(\mathcal{A}) \cap T) = (f^{-1})^{-1} \left(\mathrm{supp}(\mathcal{A})\right)$.

In general, $\mathrm{supp}(X) = \mathrm{supp}(\mathsf{loc}(X))$ for all $X \in D(R)$. Because $f_\bullet(\mathsf{loc}(X)) = \mathsf{loc}(f_\bullet X)$, the result holds with X in place of $\mathcal{A}$. $\square$

Corollary 6.2. *For all X in $D(R)$,*

$$supp(i_\bullet f_\bullet X) = supp(X) \cap T.$$

Proof. Using Propositions 5.12 and 6.1 we compute

$$\operatorname{supp}(i_\bullet f_\bullet X) = f^{-1}(\operatorname{supp}(f_\bullet X)) = f^{-1}\left(f(\operatorname{supp}(X) \cap T)\right) = \operatorname{supp}(X) \cap T.$$

□

Remark 6.3. In [IK11], the authors consider the general case of a tensor triangulated functor $F : \mathcal{T} \to \mathcal{U}$ between well generated tensor triangulated categories. They define support $\operatorname{Sp}(-)$ in terms of prime elements in the distributive lattice DL, and show that if F is well-defined on the level of Bousfield classes and preserves coproducts, then there is a continuous map $\operatorname{Sp}(F) : \operatorname{Sp}(\mathcal{U}) \to \operatorname{Sp}(\mathcal{T})$ such that

$$\operatorname{supp}_{\mathcal{U}}(FX) = \operatorname{Sp}(F)^{-1}(\operatorname{supp}_{\mathcal{T}}(X)) \text{ for all } X.$$

In the case where R and S are Noetherian, the two notions of support are equivalent [IK11, 6.10]. If we consider $i_\bullet : D(S) \to D(R)$, then Proposition 5.12 says that

$$\operatorname{Sp}(i_\bullet)^{-1} = f^{-1} : \mathsf{Spec}\, S \to \mathsf{Spec}\, R.$$

If we consider $f_\bullet : D(R) \to D(S)$, then Proposition 6.1 says that

$$\operatorname{Sp}(f_\bullet)^{-1} = (f^{-1})^{-1} : \mathsf{Spec}\, R \to \mathsf{Spec}\, S.$$

Example 6.4. *Take an arbitrary subset $W \subseteq \mathsf{Spec}\, S$. The localizing subcategory of $D(R)$ corresponding to $f^{-1}(W)$ is*

$$\mathsf{loc}\left(\overline{k_{f^{-1}\mathfrak{p}}} \;:\; \mathfrak{p} \in W\right).$$

Applying $f_\bullet$ to this gives $\mathsf{loc}\left(\overline{k_\mathfrak{p}} \;:\; \mathfrak{p} \in W\right)$ in $D(S)$, which corresponds to W under the classification.

Now we look at the action of $i_\bullet$ on localizing subcategories.

Proposition 6.5. *Let $\mathcal{B}$ be a localizing subcategory of $D(S)$. Then*

$$\mathit{supp}(i_\bullet \mathcal{B}) = f^{-1}\left(\mathit{supp}(\mathcal{B})\right).$$

Proof. Since R and S are Noetherian, every localizing subcategory is principal. Using fact that $\operatorname{supp}(X) = \operatorname{supp}(\mathsf{loc}(X))$ for all X, the result follows from Proposition 5.12. □

This lemma implies that, via the lattice isomorphisms $\mathbf{2}^{\mathsf{Spec}\, R} \cong \mathsf{Loc}(D(R))$ and $\mathbf{2}^{\mathsf{Spec}\, S} \cong \mathsf{Loc}(D(S))$, we have "$f^{-1} = i_\bullet$". Because $(f^{-1})^{-1} \circ f^{-1}$ is the identity on $\mathsf{Spec}\, S$, the composition $f_\bullet \circ i_\bullet$ is the identity on $\mathsf{Loc}(D(S))$. We make this more precise in the next proposition.

Proposition 6.6. *Take $\mathfrak{p} \in \mathsf{Spec}\, S$. Then*

$$\mathsf{loc}(i_\bullet \overline{k_\mathfrak{p}}) = \mathsf{loc}(\overline{k_{f^{-1}\mathfrak{p}}}), \;\; \textit{so} \;\; \langle i_\bullet \overline{k_\mathfrak{p}} \rangle = \langle \overline{k_{f^{-1}\mathfrak{p}}} \rangle.$$

Thus for all localizing subcategories $\mathcal{B}$ of $D(S)$, we have $f_\bullet i_\bullet \mathcal{B} = \mathcal{B}$.

Remark 6.7. The last statement might seem to follow from the one-to-one correspondence between localizing subcategories and Bousfield classes, in light of the fact that $\langle f_\bullet i_\bullet X \rangle = \langle X \rangle$ for all X in $D(S)$. It does, but some care is necessary; see Section 6.3.

Proof. For all $\mathfrak{q} \in U$, we have $i_\bullet \overline{k_\mathfrak{p}} \wedge \overline{k_\mathfrak{q}} = i_\bullet(\overline{k_\mathfrak{p}} \wedge f_\bullet \overline{k_\mathfrak{q}}) = i_\bullet(0) = 0$, by Lemma 5.11. For all $\mathfrak{q} \in T \backslash \{f^{-1}(\mathfrak{p})\}$, we have

$$i_\bullet \overline{k_\mathfrak{p}} \wedge \overline{k_\mathfrak{q}} = i_\bullet(\overline{k_\mathfrak{p}} \wedge f_\bullet \overline{k_\mathfrak{q}}) = i_\bullet(\overline{k_\mathfrak{p}} \wedge \overline{k_{f(\mathfrak{q})}}) = 0,$$

by Lemma 2.5. Therefore

$$\mathsf{loc}(i_\bullet \overline{k_\mathfrak{p}}) \subseteq \left\langle \coprod_{\mathfrak{q} \neq f^{-1}(\mathfrak{p})} \overline{k_\mathfrak{q}} \right\rangle = \mathsf{loc}\left(\overline{k_{f^{-1}\mathfrak{p}}}\right).$$

On the other hand,

$$i_\bullet \overline{k_\mathfrak{p}} \wedge \overline{k_{f^{-1}\mathfrak{p}}} = i_\bullet(\overline{k_\mathfrak{p}} \wedge \overline{k_\mathfrak{p}}) \neq 0,$$

because $i_\bullet$ is injective, so $i_\bullet \overline{k_\mathfrak{p}} \neq 0$ and $\mathsf{loc}(i_\bullet \overline{k_\mathfrak{p}})$ is non-zero. Because $\mathsf{loc}(\overline{k_{f^{-1}\mathfrak{p}}})$ is a minimal non-zero localizing subcategory, we have equality.

The last statement follows from the classification of localizing subcategories and Proposition 3.4. □

6.2. **Thick subcategories.** When R is Noetherian we can strengthen the results of Section 5.1.

Lemma 6.8. *The function $f^{-1} : \mathsf{Spec}\, S \to \mathsf{Spec}\, R$ takes specialization-closed subsets to specialization-closed subsets.*

Proof. A set of prime ideals is specialization-closed if and only if it is the union of closed subsets. We know that $f^{-1}(\cup W_\alpha) = \cup f^{-1}(W_\alpha)$. Because f is surjective, the map f^{-1} is a homeomorphism onto its image, thus closed. □

We have the following improvement on Proposition 5.21.

Proposition 6.9. *Let $\mathcal{B}$ be a thick subcategory of finite objects in $D(R)$. Then*

$$f^{-1}(supp(f_\bullet \mathcal{B})) = supp(\mathcal{B}) \cap T.$$

Proof. Because R and S are Noetherian, all thick subcategories of finite objects are principal. The result follows from Proposition 6.1. □

Lemma 6.10. *Given $\mathfrak{p} \in \mathsf{Spec}\, S$, the objects $f_\bullet(R/f^{-1}\mathfrak{p})$ and $S/\mathfrak{p}$ generate the same thick subcategory.*

Proof. Since $f^{-1}\mathfrak{p}$ is finitely generated, $R/f^{-1}\mathfrak{p}$ is defined, and is finite. Therefore it suffices to show that $f_\bullet(R/f^{-1}\mathfrak{p})$ and $S/\mathfrak{p}$ have the same support. We know $\mathrm{supp}(S/\mathfrak{p}) = V(\mathfrak{p})$, and use Proposition 6.1 to calculate

$$\mathrm{supp}(f_\bullet(R/f^{-1}\mathfrak{p})) = f(\mathrm{supp}(R/f^{-1}\mathfrak{p}) \cap T)$$

$$= f(V(f^{-1}\mathfrak{p}) \cap T) = f(f^{-1}(V(\mathfrak{p}))) = V(\mathfrak{p}).$$

□

6.3. **Bousfield lattices.** With the Noetherian rings R and S, there is a correspondence between the lattice of localizing subcategories and the Bousfield lattice:

$$\mathsf{loc}\left(\overline{k_{\mathfrak{q}}}\ :\ \mathfrak{q} \in W\right) \longleftrightarrow \left\langle \bigvee_{\mathfrak{q}\in W^c} \overline{k_{\mathfrak{q}}} \right\rangle.$$

Some caution is required, however, because we have defined different operations $i_\bullet$ on Bousfield classes and on localizing subcategories, and these do not always agree.

For example, as a Bousfield class, we have $i_\bullet : \langle 0\rangle \mapsto \langle i_\bullet 0\rangle = \langle 0\rangle$. But

$$\langle 0\rangle = \mathsf{loc}(S) = \mathsf{loc}\left(\coprod_{\mathfrak{q}\in \mathsf{Spec}\, S} \overline{k_{\mathfrak{q}}}\right),$$

and, in light of Proposition 6.6, as a localizing subcategory we have

$$i_\bullet : \mathsf{loc}\left(\coprod_{\mathfrak{q}\in \mathsf{Spec}\, S} \overline{k_{\mathfrak{q}}}\right) \mapsto \mathsf{loc}\left(\coprod_{\mathfrak{q}\in \mathsf{Spec}\, S} i_\bullet \overline{k_{\mathfrak{q}}}\right) = \mathsf{loc}\left(\coprod_{\mathfrak{p}\in T} \overline{k_{\mathfrak{p}}}\right) = \left\langle \bigvee_{\mathfrak{p}\in U} \overline{k_{\mathfrak{p}}} \right\rangle.$$

This is not surprising, because as an operation on Bousfield classes or on localizing subcategories, $i_\bullet$ is not surjective. The difference is always precisely $\langle \bigvee_{\mathfrak{p}\in U} \overline{k_{\mathfrak{p}}}\rangle = \mathsf{loc}\left(\coprod_{\mathfrak{p}\in T} \overline{k_{\mathfrak{p}}}\right)$.

On the other hand, because $f_\bullet$ is surjective (Proposition 3.4 and Lemma 5.4), its two defined operations always agree.

When R is Noetherian, the lattice bijection $\mathsf{BL}_{D(R)}/J \xrightarrow{\sim} \mathsf{BL}_{D(S)}$ is actually a splitting of the Bousfield lattice $\mathsf{BL}_{D(R)}$.

Lemma 5.11 implies that

$$J = \left\{ \langle X\rangle \mid \langle X\rangle \le \left\langle \coprod_{\mathfrak{q}\in U} \overline{k_{\mathfrak{q}}} \right\rangle \right\}.$$

Because $\mathsf{BL}_{D(R)}$ is just a Boolean algebra on the classes $\langle\overline{k_{\mathfrak{q}}}\rangle$ for $\mathfrak{q} \in \mathsf{Spec}\, R$, and we have the partition $\mathsf{Spec}\, R = T \coprod U$, we get that

$$\mathsf{BL}_{D(R)} \cong \mathsf{BL}/J \times J \cong \mathsf{BL}_{D(S)} \times J.$$

Remark 6.11. In Proposition 6.12 of [IK11], the authors show that any smashing localization functor gives a splitting of the Bousfield lattice. If we take $L : D(R) \to D(R)$ to be finite localization at $\mathsf{th}(R/f^{-1}\mathfrak{p}\ :\ \mathfrak{p} \in \mathsf{Spec}\, S)$, then the support of this thick subcategory is T, which is specialization-closed. The L-acyclics are

$$\mathsf{loc}\left(\coprod_{\mathfrak{q}\in T} \overline{k_{\mathfrak{q}}}\right) = \left\langle \coprod_{\mathfrak{q}\in U} \overline{k_{\mathfrak{q}}} \right\rangle,$$

and L is smashing since finite. The splitting from [IK11, 6.12] is exactly the same as the one above induced by $f_\bullet$, described above.

7. The specific map $f : \Gamma \to \Lambda$

In this section, we consider a specific example where f is surjective but neither R nor S is Noetherian. As in Section 5.3, fix $n_i > 1$ and a prime p, and define

$$\Gamma := \frac{\mathbb{Z}_{(p)}[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)} \text{ and } \Lambda := \frac{\mathbb{F}_p[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}.$$

Let $f : \Gamma \to \Lambda = \Gamma/p\Gamma$ be the surjective projection map. Note that $\Gamma \xrightarrow{p} \Gamma \to i_\bullet\Lambda$ is an exact triangle in $D(\Gamma)$, and the object $i_\bullet\Lambda$ can be represented in $D(\Gamma)$ as the finite object

$$\left(\cdots \to 0 \to \Gamma \xrightarrow{p} \Gamma \to 0 \to \cdots\right).$$

Proposition 7.1. *For all X in $D(\Lambda)$, we have $\langle f_\bullet i_\bullet X\rangle = \langle X\rangle$.*

Proof. The ring Λ is local, so every X in $D(\Lambda)$ can be represented by a free resolution

$$\cdots \to \coprod \Lambda \to \coprod \Lambda \to \coprod \Lambda \to \cdots .$$

Applying $i_\bullet$, we get

$$\cdots \to \coprod i_\bullet\Lambda \to \coprod i_\bullet\Lambda \to \coprod i_\bullet\Lambda \to \cdots ,$$

which is

$$\begin{tikzcd}
\cdots \arrow[r] & \coprod \Gamma \arrow[r] \arrow[dr, "\oplus p"] & \coprod \Gamma \arrow[r] \arrow[dr, "\oplus p"] & \coprod \Gamma \arrow[r] & \cdots \\
\cdots \arrow[r] & \coprod \Gamma \arrow[r] & \coprod \Gamma \arrow[r] & \coprod \Gamma \arrow[r] & \cdots
\end{tikzcd}$$

Using $f_\bullet\Gamma = \Lambda$, and Remark 2.2, we see that $f_\bullet i_\bullet\Lambda$ in $D(\Lambda)$ is

$$\left(\cdots \to 0 \to \Lambda \xrightarrow{0} \Lambda \to 0 \to \cdots\right),$$

which is just $\Lambda \oplus \Sigma\Lambda$. Similarly, applying $f_\bullet$ to $i_\bullet X$ above, we get $f_\bullet i_\bullet X = X \oplus \Sigma X$.

This shows that $\mathsf{th}(f_\bullet i_\bullet X) = \mathsf{th}(X)$, and so $\langle f_\bullet i_\bullet X\rangle = \langle X\rangle$ for all X. □

Therefore the results of Section 4 apply. From Section 5.3, we know that $\mathsf{th}(i_\bullet\Lambda)$ is the unique nontrivial thick subcategory of finite objects in $D(\Gamma)$. Let $L : D(\Gamma) \to D(\Gamma)$ be finite localization at $\mathsf{th}(i_\bullet\Lambda)$. Then L is smashing, and $\langle L\Gamma\rangle = \mathsf{loc}(i_\bullet\Lambda)$.

Given a self-map, the cofiber and the sequential colimit form a complemented pair of Bousfield classes [HPS97, 3.6.9]. The cofiber of $\Gamma \xrightarrow{p} \Gamma$ is $i_\bullet\Lambda$. The sequential colimit $p^{-1}\Gamma$ is a module concentrated in degree zero, with zeroth homology

$$\mathsf{colim}(\Gamma \xrightarrow{p} \Gamma) = \frac{\mathbb{Q}[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)} =: \Pi.$$

The map $g : \Gamma \to \Pi$, coming from $\mathbb{Z}_{(p)} \hookrightarrow \mathbb{Q}$, induces functors $g_\bullet$ and $i_\bullet$ between $D(\Gamma)$ and $D(\Pi)$, and we can identify the sequential colimit $p^{-1}\Gamma$ in $D(\Gamma)$ with $i_\bullet\Pi$. Therefore in the Bousfield lattice of $D(\Gamma)$ we have

$$\langle i_\bullet\Lambda\rangle \vee \langle i_\bullet\Pi\rangle = \langle\Gamma\rangle \text{ and } \langle i_\bullet\Lambda\rangle \wedge \langle i_\bullet\Pi\rangle = \langle 0\rangle.$$

Because the localization functor L is smashing, with acyclics $\mathsf{loc}(i_\bullet\Lambda)$, the L-locals are $\langle i_\bullet\Lambda\rangle$. Because we know that $\langle i_\bullet\Lambda\rangle$ is complemented by $\langle i_\bullet\Pi\rangle$, we can conclude that the L-acyclics are given by $\langle i_\bullet\Pi\rangle$ and the locals by $\mathsf{loc}(i_\bullet\Pi)$.

As noted in Remark 6.11, every smashing localization functor gives a splitting of the Bousfield lattice. In this context, this is

$$\mathsf{BL}_{D(\Gamma)} \xrightarrow{\sim} \mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda)} \times \mathsf{BL}_{\mathsf{loc}(i_\bullet\Pi)}, \text{ where}$$
$$\langle X\rangle \mapsto (\langle X\wedge i_\bullet\Lambda\rangle, \langle X\wedge i_\bullet\Pi\rangle)\,.$$

The inverse sends $(\langle X\rangle, \langle Y\rangle) \mapsto \langle X\rangle \vee \langle Y\rangle$.

Furthermore, Lemma 4.5 applies, and says that the complement of $\langle i_\bullet\Lambda\rangle$ is $\langle M\rangle$, the join of all classes in the kernel of $f_\bullet : \mathsf{BL}_{D(\Gamma)} \to \mathsf{BL}_{D(\Lambda)}$. Therefore

$$J = \{\langle X\rangle \mid \langle X\rangle \leq \langle i_\bullet\Pi\rangle\}.$$

It is not hard to show that in fact

$$J = \{\langle X\rangle \mid \langle X\rangle \leq \langle i_\bullet\Pi\rangle\} = \{\langle X\rangle \mid X \in \mathsf{loc}(i_\bullet\Pi)\} = \mathsf{BL}_{\mathsf{loc}(i_\bullet\Pi)}.$$

Therefore, from L we get a splitting

$$\mathsf{BL}_{D(\Gamma)} \xrightarrow{\sim} \mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda)} \times J.$$

This is consistent with Proposition 4.2, which says $f_\bullet$ induces a poset isomorphism

$$\mathsf{BL}/J = \mathsf{BL}_{\mathsf{loc}(i_\bullet\Lambda)} \xrightarrow{f_\bullet} \mathsf{BL}_{\mathsf{loc}(\Lambda)} = \mathsf{BL}_{D(\Lambda)} \text{ and so}$$
$$\mathsf{BL}_{D(\Gamma)} \cong \mathsf{BL}_{D(\Lambda)} \times J.$$

8. Still to do, before submitting

This work seems incomplete without considering an injective map $f : R \to S$, at least in the case where R and S are both Noetherian.

Most of the questions in Sections 5 and 6 seem answerable, except Question 5.6 which is vague.

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